NORMAL HOMOMORPHISMS OF FORTIFIED JOIN HYPERGROUPS

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Abstract

Here appears a study on the normal homomorphisms of the fortified Join Hypergroups. The behaviour of the c and a-elements is being studied. Special types of subhypergroups (i.e. the symmetrical and join ones) appear in certain cases of kernels and images. Also the complete homomorphism is being introduced and studied.
1. Introduction.

During his speech in the 5th AHA Gerasimos G. Massouros referred explicitly to the technological problems that led to the introduction of a non scalar neutral element in the join hypergroup, in order to be able to describe and deal with problems that appear in the Theory of Automata. Thus the consideration of such an element led to the introduction of the Fortified Join Hypergroup in the Theory of Hypercompositional Structures. This one, as well as the first new hypercompositional structures that appeared, along with a sample of their applications, have been announced during the 4th AHA [5]. Other structures, based on the Fortified Join Hypergroup, that are directly associated with the Theory of Automata, have been presented by G.G. Massouros, during his speech that was mentioned above. Today it seems that this new structure can be applied to biology as well.

This paper deals with the study of a special type of the homomorphisms of the Fortified Join Hypergroup.

Definition 1.1. The Fortified Join Hypergroup (F.J.HG.) is a Join Hypergroup $(H, +)$ (that is a commutative hypergroup in which the join axiom is being verified [9],[4]) that additionally satisfies the axioms:

\[ FJ_1 \quad (\exists 0 \in H) \quad (\forall X \in H) \quad [(X \in 0 + X) \land (0 + 0 = 0)] \]

\[ FJ_2 \quad (\forall X \in H \setminus \{0\}) \quad (\exists X' \in H \setminus \{0\}) \quad [0 \in X + X'] \quad (X' = -X). \]

It has been proved [6] that this structure consists of two kinds of elements: the canonical elements (c-elements) for which $X + 0 = X$, and the attractive elements (a-elements) for which the equality $X + 0 = \{X, 0\}$ holds. In a F.J.HG. $(H, +)$ the set of the c-elements is denoted by $CH$ and the set of the a-elements by $AH$. The equality $-(x - x) = x - x$ is not always valid in a F.J.HG. This gives another classification of the F.J.HG. elements into the normal that satisfy it, and the abnormal ones [6].

The study of the F.J.HG. subhypergroups showed that two special kinds of subhypergroups, the join and the symmetrical ones, present particular interest [6]. A subhypergroup $h$ of a join hypergroup is named join if for every four of its elements the join axiom is satisfied in it, that is if for every $x, y, z, w \in h$ that satisfy $(x : y) \cap (z : w) \neq \emptyset$, it derives that $(x : y) \cap (z : w) \subseteq h$ (which implies that $x : y \subseteq h$, for every $x, y \in h$). A subhypergroup $(h, +)$ of a F.J.HG. is called symmetrical if $-x \in h$, for every $x \in h$. It has been proved that every join subhypergroup of a F.J.HG. is a symmetrical one, while the opposite is not valid [6]. More details on the F.J.HG. and its...
properties, some of which will be used in the following paragraph can be found in this volume in [8]. The extended study of this structure and some of its applications appear in [6].

In this study we mainly refer to the normal (or good) homomorphisms [1]. According to the terminology introduced by M.Krasner [2], if $H$ and $H'$ are two hypergroups, then a homomorphism from $H$ to $H'$ is a mapping $\varphi : H \to P(H')$ such that

$$\varphi(x + y) \subseteq \varphi(x) + \varphi(y), \ \forall x, y \in H.$$ 

A homomorphism is named normal if $\varphi$ is a mapping from $H$ to $H'$ and

$$\varphi(x + y) = \varphi(x) + \varphi(y), \ \forall x, y \in H.$$ 

It must be noted that for the normal homomorphisms the above equality is not valid for the induced hypercomposition. Instead of this, we have the relation: $\varphi(x : y) \subseteq \varphi(x) : \varphi(y)$ [8].

2. Homomorphisms of F.J.HG.

Let's suppose that $H$ and $H'$ are two F.J.HG. and let $\varphi$ be a normal homomorphism from $H$ to $H'$. As usual [7], we define the kernel of $\varphi$, denoted by $\ker \varphi$, to be the subset $\varphi^{-1}(\varphi(0))$ of $H$ and we denote the homomorphic image $\varphi(H)$ of $H$, with $\text{Im}\varphi$.

Proposition 2.1.

(i) If $x \in C_H \cap \ker \varphi$, then $-x \in \ker \varphi$, and $A_H \subseteq \ker \varphi$.

(ii) $\ker \varphi$ is a subhypergroup of $H$.

(iii) $\text{Im} \varphi$ is a subhypergroup of $H'$, which generally does not contain the element $\varphi'(0') \in H'$, but $\varphi(0)$ is neutral element in $\text{Im} \varphi$.

(iv) If $\varphi(0) = 0$, then $\varphi(A_H) \subseteq A_H' \cup \{0\}$ and $\varphi(C_H) \subseteq C_H'$.

**Proof.** (i) It has been proved that if $x \in C_H$, then $-x \in C_H$ [6]. So for $\varphi(-x)$ we have: $\varphi(x - x) = \varphi(0) + \varphi(-x) = \varphi(0 + (-x)) = \varphi(-x)$. But $0 \in x - x$, so $\varphi(0) \in \varphi(x - x) = \varphi(-x)$ and thus $\varphi(-x) = \varphi(0)$, therefore $-x \in \ker \varphi$. Moreover it is known that for the set $A$ of the a-elements of $H$, $A \subseteq x - x$ holds for every $x \in C$ [6]. Thus $\varphi(A) \subseteq \varphi(x - x) = \varphi(0)$, hence $A \subseteq \ker \varphi$.

(ii) Let $x \in \ker \varphi$. Then $\varphi(x + \ker \varphi) = \varphi(0)$. So $x + \ker \varphi \subseteq \ker \varphi$. Next let $y \in \ker \varphi$. Then, if $-x \in \ker \varphi$, we have: $y \in (x - x) + y = x + (-x + y) \subseteq x + \ker \varphi$. If $-x \notin \ker \varphi$, then, from (i), $-x$ is an a-element, and so we have: $\varphi(-x + y) = \varphi(-x) + \varphi(0) = \varphi(-x + 0) = \varphi(\{-x, 0\}) = \{\varphi(-x), \varphi(0)\}$. 


Now let $w$ be an element of $\ker \varphi$ such that $w \in -x + y$. Since $w \neq -x$, as $-x \notin \ker \varphi$, we have that the reversibility of $-x$ holds [6] and therefore $y \in x + w$, thus $y \in x + \ker \varphi$. Consequently $\ker \varphi \subseteq x + \ker \varphi$ and so $\ker \varphi$ is a subhypergroup of $H$.

(iii) Let $x \in H$. Then $\varphi(x) + \varphi(H) = \bigcup_{w \in H} \varphi(x + w) = \varphi(x + H) = \varphi(H)$.

Thus $\text{Im}\varphi$ is a subhypergroup of $H'$. Yet we have $\varphi(0) \in \varphi(x - x) = \varphi(x) + \varphi(-x)$ and since $x \in 0 + x$ we have $\varphi(x) \in \varphi(0) + \varphi(x)$.

(iv) Let $a$ be an $a$-element, then $\varphi(a) + 0 = \varphi(a + 0) = \varphi(\{a, 0\}) = \{\varphi(a), 0\}$. If $c$ is a $c$-element, then we have: $\varphi(c) + 0 = \varphi(c + 0) = \varphi(c)$.

A homomorphism does not necessarily map $a$-elements to $a$-elements. Relatively we have the example:

**Example 2.1.** Let $(G, +)$ be an abelian group. Then $(G, +)$, where $+$ is the hypercomposition: $x + y = \{x, y, x + y\}$, is a F.J.HG, which has only $a$-elements. Next let $K$ be a totally ordered and symmetrical set with regard to a center $0 \in K$. In $K$ we can define a partition: $K = K^- \cup \{0\} \cup K^+$, such that $x < 0 < y$ for every $x \in K^-$, $y \in K^+$, $x < y \Rightarrow -y < -x$ for every $x, y \in K$, where $-x$ the symmetrical of $x$ w.r.t. 0. If in this set the hypercomposition:

\[
\begin{align*}
x + y &= \{x, y\} \quad \text{if } y \neq -x \text{ and } y \neq 0 \\
x + (-x) &= K \quad \text{for every } x \in K \\
x + 0 &= x \quad \text{for every } x \in K
\end{align*}
\]

is introduced, then $K$ becomes a canonical hypergroup [3]. Now suppose that $G$ is totally ordered and symmetrical with regard to 0 and that $|K| \geq 5$. Then, if $w, y \in K^+$, the function:

\[
\varphi(x) = \begin{cases} 
  w, & \text{if } x \in G^+ \cup \{0\} \\
  y, & \text{if } x \in G^-
\end{cases}
\]

is a normal homomorphism which maps $a$-elements to $c$-elements.

**Lemma 2.1.** If $a, a'$ are $a$-elements and $a + a' = a$, then $a = a'$.

**Proof.** $a + a' = a \Rightarrow 0 + (a + a') = 0 + a \Rightarrow \{0, a\} + a' = \{0, a\} \Rightarrow (0 + a') \cup (a + a') = \{0, a\} \Rightarrow \{0, a, a'\} = \{0, a\}$. Thus $a' \in \{0, a\}$, since $a' \neq 0$, it follows that $a' = a$.

**Proposition 2.2.** Let $\varphi$ be a normal homomorphism from $H$ to $H'$. The following are true:
(i) If the image of an a-element is a non zero c-element, then \( \text{Im} \varphi \subseteq C'_H \).

(ii) If the image of an c-element is an a-element, then it belongs to \( \ker \varphi \) and \( A_H \subseteq \ker \varphi \).

**Proof.** (i) Because of Proposition 2.1 (iv), \( \varphi(0) \neq 0 \). Let \( a \) be an a-element, \( c \) be a c-element and \( \varphi(a) \) be a c-element. Since the sum of an a-element and a c-element, different from 0, is always the participating c-element [6], we have:

\[
(a) \quad \varphi(c) + 0 = \varphi(c + a) + 0 = [\varphi(c) + \varphi(a)] + 0 = \varphi(c) + [\varphi(a) + 0] = \varphi(c) + \varphi(a) = \varphi(c + a) = \varphi(c).
\]

So \( \varphi(c) \) is a c-element.

(\( \beta \)) For \( \varphi(0) \) we have: \( \varphi(a) + 0 = \varphi(a) \Rightarrow \varphi(0) + \varphi(c) + 0 = \varphi(0) + \varphi(a) \Rightarrow \varphi(0 + a) + 0 = \varphi(0, a) + 0 = \varphi(\{0, a\}) + 0 = \{\varphi(0), \varphi(a)\} \Rightarrow [\varphi(0) + 0] \cup [\varphi(a) + 0] = \{\varphi(0), \varphi(a)\} \Rightarrow [\varphi(0) + 0] \cup \{\varphi(a)\} = \{\varphi(0), \varphi(a)\}. \) If \( \varphi(0) + 0 = \{\varphi(0), 0\}, \) then \( 0 \in \{\varphi(0), \varphi(a)\} \) which is absurd, so \( \varphi(0) \) is a c-element.

(\( \gamma \)) Now let \( b \) be an a-element. Supposing that \( \varphi(b) \) is an a-element, we have \( \varphi(b) + \varphi(0) = \varphi(0) \), because \( \varphi(0) \) is a c-element. But \( \varphi(b) + \varphi(0) = \varphi(b, 0) = \{\varphi(b), \varphi(0)\}. \) Thus \( \varphi(0) = \{\varphi(b), \varphi(0)\} \) which means that \( \varphi(0) = \varphi(b). \) But this is absurd, since it has been proved, in (\( \beta \)), that \( \varphi(0) \) is a c-element, so \( \varphi(b) \) is a c-element.

(ii) Let \( a \) be an a-element and \( c \) be a c-element whose image \( \varphi(c) \) is an a-element. Then \( \varphi(a) + \varphi(c) = \varphi(a + c) = \varphi(c). \) First we observe that \( \varphi(a) \neq 0 \) because \( \varphi(c) \) is an a-element and the sum of an a-element with 0 is the participating a-element and the 0. Also \( \varphi(a) \) is not a c-element, because if it were, then according to (i) all the elements of \( \text{Im} \varphi \) should have been c-elements. Thus \( \varphi(a) \) is an a-element and so according to Lemma 2.1 \( \varphi(a) = \varphi(c). \) So \( \varphi(a) = \varphi(c) \Rightarrow \varphi(a) + \varphi(0) = \varphi(c) + \varphi(0) \Rightarrow \varphi(a + 0) = \varphi(c + 0) \Rightarrow \varphi(\{a, 0\}) = \varphi(c) \Rightarrow \{\varphi(a), \varphi(0)\} = \varphi(c). \) Therefore \( \varphi(a) = \varphi(0) = \varphi(c), \) so \( c \in \ker \varphi \) and because of Proposition 2.1 (i) all the a-elements belong to \( \ker \varphi \).

**Remark 2.1.** The fact that \( z \) is an a-element and \( z \in \ker \varphi \) does not imply that \( -z \) belongs to \( \ker \varphi \), as it is being demonstrated in the Example that follows. So, even though \( \ker \varphi \) is a subhypergroup of \( H \), generally it is not a symmetrical subhypergroup of \( H \). But from the following example it follows that \( \text{Im} \varphi \) is generally not a symmetrical subhypergroup of \( H' \) either, even in the case that \( \varphi(0) = 0 \).
Example 2.2. Let $H$ be a totally ordered and symmetrical set with regard to a center $0 \in H$, in which we consider the partition $H = H^- \cup \{0\} \cup H^+$ and let "+" be the hypercomposition:

$$x + y = \{x, y\} \text{ if } y \neq -x;$$

$$x + (-x) = [0, |x|] \cup \{-|x|\}; \; 0 + 0 = 0,$$

where $|x| = x$, if $x \in H^+$, $-x$, if $x \in H^-$ and $0$, if $x = 0$. Then, $(H,+)$ becomes a F.J.HG. and every $x \in H \setminus \{0\}$ is an $a$-element. Next we consider the function $\varphi : H \rightarrow H$ defined as follows:

$$\varphi(x) = \begin{cases} 0, & \text{if } x \in H^+ \cup \{0\} \\ x, & \text{if } x \in H^- \end{cases}.$$

Then $\varphi$ is a homomorphism from $H$ to $H$. For this homomorphism we have $\ker \varphi = H^+ \cup \{0\}$, $\varphi(0) = 0$ and $\text{Im} \varphi = H^- \cup \{0\}$.

From the above we are led to the Definition:

Definition 2.1. A homomorphism for which $x \in \ker \varphi$ implies that $-x \in \ker \varphi$, will be called a complete homomorphism.

Proposition 2.3. Let $\varphi$ be a normal homomorphism between $H$ and $H'$. Then:

(i) if $\varphi$ is surjective, it is complete and $\varphi(0) = 0$.

(ii) if $\ker \varphi = \{0\}$ and $\varphi$ is surjective, $\varphi$ is an isomorphism.

(iii) if $\ker \varphi \cap CH \neq 0$, $\varphi$ is complete.

Proof. (i) Let $\varphi$ be a normal epimorphism between the F.J.HG. $H$ and $H'$. Firstly we shall prove that $\varphi(0) = 0$. Indeed, since $\varphi$ is epimorphism then for the $-\varphi(0)$ there will exist $x \in H$ such that $\varphi(x) = -\varphi(0)$. Consequently we have: $0 \in -\varphi(0) + \varphi(0) \Rightarrow 0 \in \varphi(x) + \varphi(0) \Rightarrow 0 \in \varphi(x + 0) \Rightarrow 0 \in \varphi(\{x,0\}) \Rightarrow 0 \in \{\varphi(x), \varphi(0)\}$. So either $\varphi(0) = 0$ or $\varphi(x) = 0$ from where $-\varphi(0) = 0$, and thus $\varphi(0) = 0$. Next let $\varphi(x) = 0$ and $\varphi(-x) \neq 0$. Then for $-\varphi(-x)$ there exists an element $b$ of $H$ which does not belong to $\ker \varphi$ such that $-\varphi(-x) = \varphi(b)$. Thus: $0 \in \varphi(x) + \varphi(b) = \varphi(-x + b)$ and therefore $(-x + b) \cap \ker \varphi \neq 0$. So let $w$ be an element of $\ker \varphi$ such that $w \in -x + b$. But according to the hypothesis $-x \notin \ker \varphi$ thus in the above relation the reversibility of $-x$ holds [6] and so $b \in w + x$, that is $b \in \ker \varphi$, which is absurd. Thus $\varphi(-x) = 0$ and therefore $\varphi$ is complete.

(ii) Let $\varphi$ be a normal epimorphism with $\ker \varphi = \{0\}$. According to (i), $\varphi$ is complete and $\varphi(0) = 0$. Thus we have: $0 \in \varphi(x - x) = \varphi(x) + \varphi(-x)$. But
since \( \ker \varphi = \{0\} \) we have \( \varphi(z), \varphi(-z) \neq 0 \), and therefore \( \varphi(-z) = -\varphi(z) \). Next let \( \varphi(x) = \varphi(y) \), then: \( 0 \in \varphi(x) - \varphi(y) = \varphi(x) + \varphi(-y) = \varphi(x - y) \) and since \( \ker \varphi = \{0\} \) we have \( 0 \in x - y \). Thus \( x = y \).

(iii) According to Proposition 2.1 (i), if the kernel of a normal homomorphism contains a \( c \)-element then it will also contain its opposite as well as all the \( a \)-elements of the F.J.H.G. Therefore for every \( x \in \ker \varphi \) we will also have \( -x \in \ker \varphi \).

For complete and normal homomorphisms the equality \( \varphi(0) = 0 \) is not generally valid. But when it is valid we have the Proposition:

**Proposition 2.4.** Let \( \varphi \) be a complete and normal homomorphism for which \( \varphi(0) = 0 \). Then \( \varphi(-z) = -\varphi(x) \) and the \( \text{Im} \varphi \) is a symmetrical subhypergroup of \( H' \).

**Proof.** From Proposition 2.1(iii) it is known that \( \text{Im} \varphi \) is a subhypergroup of \( H' \) and since \( \varphi(0) = 0 \) we have that \( 0 \in \text{Im} \varphi \). Now let \( \varphi(x) \) be an arbitrary element of \( \text{Im} \varphi \). Then we have: \( 0 = \varphi(0) \in \varphi(x-x) = \varphi(x) + \varphi(-x) \).

Since \( \varphi \) is a complete homomorphism, if \( \varphi(x) \neq 0 \) then \( \varphi(-x) \neq 0 \) as well. Thus from the above relation it follows that \( \varphi(-x) = -\varphi(x) \) and so the Proposition.

**Corollary 2.1.** Let \( \varphi \) be a complete and normal homomorphism for which \( \varphi(0) = 0 \), and let \( h \) be a symmetrical subhypergroup of \( H \). Then \( \varphi(h) \) is a symmetrical subhypergroup of \( H' \).

**Proposition 2.5.** Let \( \varphi \) be a normal homomorphism. Then the set \( \ker \varphi = -\varphi^{-1}(\varphi(0)) \cup \varphi^{-1}(\varphi(0)) \) is a symmetrical subhypergroup of \( H \).

**Proof.** \( 0 \) belongs to \( \ker \varphi \) and if \( x \in \ker \varphi \) then \( -x \in \ker \varphi \). Let \( x, y \in \ker \varphi \).

\( \alpha \) If \( x, y \in \ker \varphi \) then, because of the Proposition 2.1 (ii), \( x + y \subseteq \ker \varphi \).

\( \beta \) If \( x, y \in \ker \varphi \) then if \( y = -x \) we have \( -x, x \in \ker \varphi \) and so \( x - x \subseteq \ker \varphi \). If \( y \) is different from \( -x \), then the relation \( x + y = -(x - y) \) is always valid [6]. But \( -x, -y \in \ker \varphi \), thus \( -x - y \subseteq \ker \varphi \), therefore \( -(x - y) \subseteq -\ker \varphi \).

\( \gamma \) If \( x \in \ker \varphi \) and if \( y \) is an element of \( -\ker \varphi \) which does not belong to \( \ker \varphi \) then \( \varphi(x + y) = \varphi(0) + \varphi(y) = \varphi(0 + y) \subseteq \varphi(\{0, y\}) = \varphi(y) + \varphi(-z) = \varphi(y) + \varphi(-z) \Rightarrow \varphi(x - z) = \varphi(y - z) \Rightarrow y - z \cap \ker \varphi \neq \emptyset \). So there exists \( w \in \ker \varphi \) such that \( \omega \in y - z \). Since \( y \notin \ker \varphi \) the
 reversibility of \( y \) holds. Thus \(-z \in -y + w\). But \(-y \in \ker \varphi\) and so \(-z \in \ker \varphi\), thus \(z \in \ker \varphi \) and therefore \(x + y \subseteq \ker \varphi \cup \ker \varphi\). So if \(x\) and \(y\) are two arbitrary elements from \([\ker \varphi]\) then \(x + [\ker \varphi] \subseteq \) \(\subseteq [\ker \varphi]\) and \(y \in [x - x] + y = x + (-z + y) \subseteq x + [\ker \varphi]\), that is \([\ker \varphi]\) \(\subseteq x + [\ker \varphi]\). Therefore \(x + [\ker \varphi] = [\ker \varphi]\), and so the Proposition.

**Corollary 2.2.** Let \(\varphi\) be a complete and normal homomorphism with \(\varphi(0) = 0\). Then \(\ker \varphi\) is a symmetrical subhypergroup of \(H\).

Generally \([\ker \varphi]\) is not a join subhypergroup, as it is being demonstrated in the Example:

**Example 2.3.** Consider the dilated \(B\)-hypergroup \(H = \{0, a_1, a_2, a_3, a_4\}\) with neutral element \(0\) [6]. We define the mapping \(\varphi : H \to H\) as follows:

\[
\varphi(x) = \begin{cases} 
0, & \text{if } x \in \{0, a_1, a_2\} \\
\ y, & \text{if } x \in \{a_3, a_4\}.
\end{cases}
\]

\(\varphi\) is a complete and normal homomorphism the kernel of which is \([\ker \varphi] = = \{0, a_1, a_2\}\). This subhypergroup is not a join one, since we have for instance \(a_1 : a_1 = H\) which is different from \([\ker \varphi]\).

**Proposition 2.6.** Let \(H, H'\) be two F.J.HGC. and let \(\varphi\) be a normal epimorphism from \(H\) to \(H'\). If \(h\) is a join subhypergroup of \(H\) which contains the kernel of \(\varphi\), then \(h' = \varphi(h)\) is a join subhypergroup of \(H'\).

**Proof.** According to Proposition 2.3 (i), \(\varphi\) is a complete homomorphism with the property \(\varphi(0) = 0\). Therefore from Corollary 2.1 it follows that \(h'\) is a symmetrical subhypergroup. Now let \(y \in \varphi(x_1) : \varphi(x_2)\) with \(x_1, x_2 \in h\). Then, since \(\varphi\) is an epimorphism, there exists an \(x \in H\) such that \(\varphi(x) = y\). From the relation \(y \in \varphi(x_1) : \varphi(x_2)\) it follows that \(\varphi(x_1) \in y + \varphi(x_2)\), that is \(\varphi(x_1) \in \varphi(x) + \varphi(x_2)\) and so \(\varphi(x_1) \in \varphi(x + x_2)\) (\(*\)). Next we distinguish between the cases:

(i) \(x_1, x_2 = 0\). Then, from (\(*\)) we have, \(\varphi(0) = \varphi(x + 0)\). If \(x\) is a \(c\)-element, then \(\varphi(0) = \varphi(x)\) and so \(\varphi(x) \in h'\). If \(x\) is an \(a\)-element, then \(0 + x + 0 \Rightarrow x \in 0 : 0\), but \(0 : 0 \subseteq h\), thus \(x \in h\), and so \(\varphi(x) \in \varphi(h) = h'\).

(ii) \(x_1 = 0, x_2 \neq 0\). Then from (\(*\)) we have \(\varphi(0) = \varphi(x + x_2)\). Thus there exists \(a \in x + x_2\), such that \(\varphi(0) = \varphi(a)\). So \(a \in \ker \varphi\) and therefore \(a \in h\). So we have: \(a \in x + x_2 \Rightarrow x \in a : x_2 \Rightarrow x \in h : x_2 \Rightarrow x \in h \Rightarrow \varphi(x) \in \varphi(h) = h'\).
(iii) $x_1 \neq 0, x_2 = 0$. Then from (ii) we have $\varphi(x_1) = \varphi(x + 0)$. If $x$ is a $c$-element then $\varphi(0) = \varphi(x)$ and so $\varphi(x) \in h'$. If $x$ is a $c$-element, then $0 \in x + 0 \Rightarrow x \in 0 : 0$, thus $x \in h$ and so $\varphi(x) \in \varphi(h) = h'$.

(iv) $x_1 \neq 0, x_2 \neq 0$. From (i) it follows that there exists $a \in x + x_2$ such that $\varphi(a) = \varphi(x_1)$. From this equality, we have: $0 \in \varphi(a) - \varphi(x_1) \Rightarrow 0 \in \varphi(a - x_1) \Rightarrow (a - x_1) \subseteq \ker \varphi \neq \emptyset$.

Let $b \in (a - x_1) \cap \ker \varphi$. If we suppose that $b \neq -x_1$, then from the relation $b \in (a - x_1)$, and since the reversibility of $-x_1$ is valid, it follows that $a \in b + x_1$, that is we have $a \in \ker \varphi + x_1 \subseteq h$. But from the relation $a \in x + x_2$ we have that $x \in a : x_2$ and so $x \in h : x_2 \subseteq h$. Thus $y = \varphi(x) \in \varphi(h) = h'$. Moreover if we suppose that $b = -x_1$, then $\varphi(-x_1) = 0$ and since $\varphi$ is complete we will have that $\varphi(x_1) = 0$ from where, since $\varphi(x_1) = \varphi(a)$, it follows that $\varphi(a) = 0$ and so $a \in \ker \varphi$. Hence for $x$ we will have $x \in a : x_2 \subseteq \ker \varphi : x_2 \subseteq h : x_2 \subseteq h$ thus again $y = \varphi(x) \in \varphi(h) = h'$, and so the Proposition.

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