AUTOMATA – LANGUAGES
&
HYPERCOMPOSITIONAL STRUCTURES

DOCTORAL THESIS
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ABSTRACT

In this dissertation the Theory of Hypercompositional Structures is being introduced into the Theory of Automata and Languages. Beyond the already known hypercompositional structures such as the join hypergroup, that in a natural way appear into the Theory of Automata and Languages, other, new ones (like the fortified join hypergroup, the hyperringoid, the hypermoduloid etc.), come into being, for the first time, and they are being studied as well.

More specifically, this dissertation consists of five chapters. The First Chapter proves the association of the theory of languages with the theory of hypercompositional structures. Indeed, in the theory of languages the
expression $\chi + \psi$ is used, where $\chi$, $\psi$ are words over an alphabet $A$, to state "either $\chi$ or $\psi$". Thus, beginning with the fact that $\chi + \psi$ is basically a biset, the set of the words $A^*$ over an alphabet $A$, endowed with the hypercomposition $w_1 + w_2 = (w_1, w_2)$ becomes, as it is being proved here, a join hypergroup. Furthermore, combining the above with the fact that $A^*$ is a semigroup with regard to the concatenation of the words, $A^*$ is endowed with a new hypercompositional structure, the hyperringoid.

A hyperringoid is a non void set $Y$ with an operation "." and a hyperoperation "+" that satisfy the axioms: [Def.III.1.1. p.94]

i. $(Y, +)$ is a hypergroup

ii. $(Y, .)$ is a semigroup

iii. the operation is bilaterally distributive to the hyperoperation.

If $(Y, +)$ is a join hypergroup, then the hyperringoid is called join hyperringoid.

The special join hypergroup which derives in this way from the theory of languages, is named $B$-hypergroup, and the respective hyperringoid, $B$-hyperringoid. In particular, in the first paragraph of this chapter, apart from the connection of the languages to $B$-hypergroups, the general properties of $B$-hypergroups are being studied.

The join sub-hypergroups are being introduced and studied in the second paragraph. A join sub-hypergroup is a sub-hypergroup of a join hypergroup that satisfies the join axiom inside it [Def.I.2.1. p.14].
In the third paragraph of this chapter the homomorphic relations, the homomorphic equivalence relations are being introduced and studied along with the several types of homomorphisms.

In the Second Chapter, starting with the notion of the empty set of words from the theory of languages, the join hypergroup is being enriched with axioms and so a new hypercompositional structure is being introduced. Actually, the use of the "null word" has led to the introduction of a non scalar neutral element in the join hypergroup with regard to which, every element has a unique opposite. Thus the fortified join hypergroup was defined. This new hypercompositional structure \((H, +)\) satisfies the axioms:

\[ \text{Def. II.1.1, p.37} \]

\text{FJ1} There exists a unique neutral element, denoted \(0\), -the zero element of \(H\)- such that for every \(x \in H\):
\[ x + 0 = x \quad \text{and} \quad 0 + 0 = 0 \]

\text{FJ2} For every \(x \in H \setminus \{0\}\) there exists one and only one element \(x' \in H \setminus \{0\}\) -opposite or symmetrical of \(x\)- denoted by \(-x\), such that
\[ 0 = x + x' \]

Especially for the case of languages, the fortified join hypergroup which corresponds to them and which motivated the development of this new structure is the dilated B-hypergroup. In this hypergroup every element is self opposite. More precisely the hypercomposition of this structure is being defined in the following way:
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\[ \begin{align*}
\rho(x, y) & \text{ if } x \neq y \\
\lambda \{0, x\} & \text{ if } x = y
\end{align*} \]

The second chapter consists of four paragraphs. The first paragraph deals with the relation of the fortified join hypergroups to the theory of languages, and examines several of their fundamental properties. Among them it is worth mentioning that, as it is being proved, they consist of two kinds of elements, the canonical and the attractive elements, that is, those elements for which \( \chi + 0 = \chi \) (c-elements), or \( \chi + 0 = [\chi, 0] \) (a-elements) holds respectively. It is especially being notified that the dilated \( B \)-hypergroup consists only of a-elements.

In the beginning of the next paragraph the property of reversibility is studied, and it is being proved partially valid. Other properties follow. It is remarkable that it is being proved that the equality \( -(\chi-\chi) = \chi-\chi \) does not hold in general and therefore two kinds of elements are being defined: the normal for which the above equality is valid and the abnormal for which it is not valid. It is being proved that the c-elements are normal, while the dilated \( B \)-hypergroup consists only of normal elements.

The third paragraph refers to the sub-hypergroups of the fortified join hypergroup, and especially to the join and the symmetrical ones. The definition of the join sub-hypergroup has been given above, while symmetrical is a sub-hypergroup which contains the opposite of every one of its elements. Among other properties it is being proved
that every join sub-hypergroup is symmetrical and that the minimum join sub-hypergroup is the sub-hypergroup which consists of all the a-elements and the 0 element (and it is equal to 0:0). Therefore the fortified join hypergroups that contain only a-elements do not have proper join sub-hypergroups. Also it is being proved that symmetrical sub-hypergroups which are not join, consist only of a-elements.

The fourth paragraph contains the monogene symmetrical sub-hypergroups. Among the aspects being studied are the definitions of the order and the attached order. The properties of all these sub-hypergroups are being studied in depth in these two paragraphs.

The Third Chapter refers to the hyperringoids. As it has already been mentioned, the set of the words over an alphabet A is a monoid with regard to the concatenation of the words. Beyond that, it is being proved that this operation is bilaterally distributive as for the hypercomposition defined in the set of words. Thus in a natural way, there derived the multiplicative-hyperadditive structure which was named hyperringoid. The hyperringoid of all the words is a special, join hyperringoid which was named B-hyperringoid or dilated B-hyperringoid, depending on the additive hypergroup which is being used. This chapter consists of six paragraphs.

The first paragraph refers to the relation between hyperringoids and languages and also to some fundamental
properties of this structure.
In the second paragraph the fortified hyperringoids are being studied, that is those hyperringoids in which the additive part is a fortified join hypergroup. These structures present many and interesting properties, one of them being that the equality \((-\chi)(-\psi) = \chi\psi\) is generally not valid in them.

The third paragraph refers to the characteristic of the hyperringoids. It is being proved that every proper normal fortified hyperringoid without divisors of 0, is of characteristic 1 and therefore every dilated B-hyperringoid is of characteristic 1.

The fourth paragraph refers to the sub-hyperringoids. One of the conclusions that have been reached here is that the set of the \(a\)-elements and 0, form a bilaterally join hyperideal which is the minimum join sub-hyperringoid of the fortified hyperringoid and that every symmetrical sub-hyperringoid is a subset of the minimum join sub-hyperringoid.

The fifth paragraph refers to the homomorphic relations, which have significant applications to the deductions we arrive at, in the fifth chapter.

Finally in the sixth paragraph elementary equations, systems and "inequalities" are being solved and the notion of the rational subset of a hyperringoid is being introduced.

The Fourth Chapter, which consists of two paragraphs refers to other hypercompositional structures that are in close relation to the theory of automata. In the first paragraph
the join polysymmetrical hypergroup (J.P.H) is being defined and studied. A J.P.H. is a join hypergroup that additionally satisfies the axioms: [Def.IV.1.1, p.148]

JP1: There exists a unique neutral element, denoted 0, -the zero element of H- such that for every \( x \in H \):
\[
x \cdot x + 0 \quad \text{and} \quad 0 + 0 = 0
\]

JP2: For every \( x \in H \setminus \{0\} \) there exists at least one element \( x' \in H \setminus \{0\} \)-opposite or symmetrical of \( x \)- denoted by \(-x\), such that: \( 0 \in x + x' \)

Here appear several of its fundamental properties while it is being proved that there exist three kinds of J.P.H. regarding the reversibility. Furthermore examples are given for each kind.

The second paragraph refers to sets with operators and hyperoperators from hyperringoids. It is being proved that if in a set \( M \) the set of operators forms a hyperringoid, then \( M \) can be endowed with a certain hypercomposition and thus acquire the structure of the hypergroup. Therefore the set of the states of an automaton acquires the structure of a certain hypergroup, since the operators of the set of states is a hyperringoid, the \( B \)-hyperringoid of the language. Next the notions of the hypermoduloid and the supermoduloid are being defined, i.e. [Def.IV.2.3, p. 162] if \( M \) is a hypergroup and \( Y \) is a hyperringoid of operators over \( M \) such that:

i. \((s + t)\lambda = s\lambda + t\lambda\)

ii. \(s(\lambda + \kappa) = s\lambda + s\kappa\)

iii. \(s(\lambda\kappa) = (s\lambda)\kappa\)
for every $k, \lambda \in Y$ and $s, t \in M$, then $M$ is named hypermoduloid over $Y$. In the case that $Y$ is a set of hyperoperators, $M$ is called supermoduloid. If $Y$ is a fortified hyperringoid and $M$ a fortified join hypergroup, then $M$ is called join hypermoduloid, resp. join supermoduloid if in addition to the above, the next axiom holds:

$$\text{iv. } s^0 = 0, \text{ resp. } iv'. 0 \in s^0$$

Also the notion of the $(s,F)$-acceptable subset of the set of operators from $M$, is being introduced and the properties of such sets are being studied. Thus [Def.IV.2.4, p.165] a subset $L$ of $Y$ becomes $(s,F)$-acceptable from $M$, (or simply is $(s,F)$-acceptable) if there exists $s \in M$ and $F \subseteq M$ (in the case of external operation) or $F \subseteq P(M)$ (in the case of external hyperoperation), such that: $\varphi s^{-1}(F) = L$, where $\varphi s$ is a function from $Y$ to $M$ (or $P(M)$ in the case of hyperoperation) for which $\varphi s(\lambda) = s\lambda.$

It worth mentioning that, as it has been proved here, if $M$ is a finite set with $\text{card} M = n$ and $Y$ a $B$-hyperringoid then for any given $F$ the acceptable from $M$ subsets of $Y$ are the solution of an $n \times n$ system which is being defined and solved here.

In the Fifth Chapter, after a brief review of all the conclusions that have been reached up to this point on the theory of languages and automata, several aspects of this theory are being viewed through the theory of hypercompositionial structures and are being proved with the use of tools and methods developed in the previous chapters.
More precisely the fifth chapter consists of four paragraphs.

The first paragraph deals with a special kind of B-hyperringoid, the linguistic hyperringoid. Every language is a subset of a linguistic hyperringoid and therefore the study of its properties can be achieved through the study of the corresponding hyperringoid. Thus in this paragraph appear results regarding a relation of great importance in the theory of languages, the equivalence of length. Next, the significance of the null word in automata is being mentioned, when and where it appears and what derives from it. Also the extent of its involvement in the formation of a rational subset of the hyperringoid of operators which is (s,F)-acceptable from a finite set M, and thus in the definition of the language that is acceptable by an automaton is being shown.

The second paragraph refers to the relation of the hypercompositional structures that have already been introduced, to systems endowed with internal memory and external inputs of data. The set of the conditions that cause such a system to move from state to state can acquire the structure of a linguistic hyperringoid and this leads to several interesting results. As an example of a linguistic hyperringoid we present the linguistic hyperringoid which is defined by a JK flip-flop. On the other hand, as it has already been proved in the previous chapters, quotients of hyperringoids with homomorphic equivalence relations define hypermoduloids. Thus a counter, which is the quotient of the
linguistic hyperringoid of the natural numbers by the equivalence relation mod n is presented as an example. Next, using conclusions that have been reached in the previous chapters, the theorem of Nerode is being proved. In chapter IV has been proved that a set M with operators from a hyperringoid Y, can be endowed, using its operators, with the structure of the hypergroup. In the paragraph which follows, considering M to be the set of states of an automaton, the attached hypergroup of the paths is being defined. Also it is being proved that certain subsets of the hyperringoid, defined through the hyperoperation in M, are rational. This approach, apart from other results, leads to the proof of the theorem of Kleene. In the last paragraph the set of states of an automaton is being endowed with several hyperoperations, which give the structure of the hypergroup to it. These attached hypergroups, describe its structure while some of them, the attached order hypergroup and the attached grade hypergroup, lead to the minimization of the given automaton. Thus, if the attached grade hypergroup is join polysymmetrical, then, based on it and according to the results developed in the previous chapters, the construction of a fortified join hypergroup leads to an automaton that accepts the same language, but has fewer states. Furthermore, through the definition of the appropriate order hypergroup, the automaton is being minimized. Next a hypergroup which describes the operation of the automaton is being defined
through the notion of the **activated element**, that is being introduced. By means of the hyperoperation being introduced, all the possible paths that the automaton can follow at any given moment during its operation can be described. The results of this hyperoperation are being calculated through a technique developed in this paragraph.