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Monogene Symmetrical Subhypergroups

of the Fortified Join Hypergroup

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The Fortified Join Hypergroup is a hypercompositional structure connected to computer theory. This paper deals with a special type of a subhypergroup of the fortified join hypergroup, which is generated by a single element. It also introduces the notion of the order \( \omega(x) \) of an element \( x \) of the fortified join hypergroup, which can be either the \( +\infty \) or a pair of a positive integer and a function.

**Keywords:** hypergroup, subhypergroup, fortified join hypergroup, languages, automata.

1. Elements from the Theory of the Fortified Join Hypergroups

The **fortified join hypergroups** came into being during the study of the theory of languages and automata with the use of methods and tools from the hypercompositional algebra [8,9,10,16]. Thus, the “sums” \( x+y \) in the theory of languages, which give as a result the biset \( \{x,y\} \), inspired the introduction of the hypercomposition \( x+y = \{x, y\} \) into the set of the words \( A^* \) over an alphabet \( A \) [7,8]. The set \( A^* \), endowed with this hypercomposition becomes a hypergroup which is a join one [7,8]. Due to its origin (the biset result of the hypercomposition), this hypergroup was named **B-hypergroup** [8,9,10].

As it is known, the join space is a hypercompositional structure, and more precisely a hypergroup, that W. Prenowitz introduced in order to study Geometry with the use of the theory of the hypergroups (e.g., see [18,19,20]). The axiom which was introduced by W. Prenowitz in a commutative hypergroup \((H, \cdot)\) and which characterizes the join space is:

\[
(a:b) \cap (c:d) \neq \emptyset \Rightarrow (ad) \cap (bc) \neq \emptyset
\]

for every \( a,b,c,d \in H \), where \( x \cdot y \) is the induced hypercomposition (division) in \( H \), that is \( x \cdot y = \{z \in H \mid x \in zy\} \) [4].
In the theory of Languages, it is also necessary to have the "null word", which is an element bilaterally absorbing as for the multiplication (concatenation of the words). This, viewed from the standpoint of the hypercompositional structures, led to an extension of the notion of the B-hypergroup as follows. Let 0 be an element not belonging to A*. Then, the following hypercomposition is defined in the set A* = A* ∪ \{0\}:

\[
\begin{align*}
x + y &= \{x, y\} \quad \text{if } x, y \in A^* \text{ and } x \neq y, \\
x + x &= \{x, 0\} \quad \text{for all } x \in A^*
\end{align*}
\]

This structure is called *dilated B-hypergroup* [8,9,10].

Yet, the above construction inspired the introduction of a non-scalar neutral element in the join hypergroup with regard to which every element has a unique opposite. Thus a new class of hypergroups, the *fortified join hypergroups*, came into being.

**Definition 1.1** A *fortified join hypergroup* (FJH) is a join hypergroup \((H, +)\) with a unique element 0, called the zero element of \(H\), such that 0+0=0, \(x \in x+0\) for all \(x \in H\) and for every \(x \in H \setminus \{0\}\) there exists one and only one element \(-x \in H \setminus \{0\}\), called the opposite of \(x\), such that 0\(\in x+(-x)\). We denote \(x+(-y)\) by \(x-y\).

This hypergroup also appears in the attached hypergroups to an automaton and yet it is involved in the procedure of the minimization of a given automaton [7,8]. A mathematical study of the fortified join hypergroup appears in [6,8,11,13,14] and several examples of such hypergroups can be found in [13,14], while a structure theorem of the FJHs can be found in [13]. Also, this hypergroup is a part of other hypercompositional structures that came into being during the study of the theory of languages and automata with the use of the hypercompositional algebra (e.g. see [15,16]). It is worth mentioning that the hypercompositional structures provide useful tools for the approach of the theory of languages and automata and that the relevant research has proved to be very interesting for both theories. For instance, J. Chalina and L. Chvalinova, working on automata without output, have succeeded to use a commutative hypergroup in such a way that they get a passage from the theory of automata with the same input alphabet into the category of commutative hypergroups and their strong homomorphisms [1,2].

The study of the FJH has revealed that all the elements of a FJH do not have the same behavior when 0 is added to themselves. The result of the addition \(x+0\) is either the singleton \(\{x\}\) or the biset \(\{0, x\}\). The elements of the first category were named *canonical* or c-elements, since they act exactly as the elements of the canonical hypergroup [17], while the others were given the name *attractive* or a-elements, because they attract "0" into the result of the hyperoperation. From now on, \(A\) will denote the set of the a-elements and \(C\) will denote the set of the c-elements.

The a and c-elements have the following fundamental properties [11,13]:
the sum of two a-elements is a subset of \( A \cup \{0\} \) and it always contains the two addends.

- the sum of two non opposite c-elements consists of c-elements, while the sum of two opposite c-elements contains all the a-elements.

- the sum of an a-element with a non zero c-element is the c-element.

Moreover, another distinction between the elements of the FJH stems from the fact that, although the equality \(-(x+y) = x - y\) holds when \( y \neq x \), it is not always valid when \( y = x \). This is shown in the following Example:

**Example 1.1.**

Let \( H \) be a totally ordered set, dense as for the order and symmetrical around a center denoted by \( 0 \in H \). With regard to this center the partition \( H = H^+ \cup \{0\} \cup H^- \) can be defined, according to which, for every \( x \in H^+ \) and \( y \in H^- \) it is \( x < 0 < y \) and \( x \leq y \implies -y \leq -x \), for every \( x, y \in H \) (where \(-x\) is the symmetrical of \( x \) with regard to 0). Then \( H \), endowed with the hypercomposition:

\[
x + y = \{x, y\}, \quad \text{if } y \neq x
\]

and

\[
x + (-x) = [0, |x|] \cup \{-|x|\}
\]

becomes a FJH in which \( x - x \neq -(x-x) \) for every \( x \neq 0 \).

In a FJH, the elements that satisfy the equality \(-(x-x) = x - x\) are called normal, while the others, are called abnormal. A FJH having only normal elements is called normal FJH, while if it contains at least one abnormal element it is called abnormal FJH.

It has been proved that all the c-elements are normal as well as that the equality \(-(x,y) = (-x)(-y)\) holds if \( y \) is normal, or if \( x \notin y \). Moreover, it has been proved that in the FJHs the reversibility holds under conditions. More precisely, \( z \in x+y \iff y \in z-x \), except if \( z=x \neq y \) in which case \( x \in x+y \iff x \in x-y \), but, generally, \( y \notin x-x \). This gives as a result that for every \( x \neq y \) it holds \( x-y = (x-y)(-y)(-x) \), while \( x-x \subseteq (x,x) \). If one of the \( x, y \) is a c-element then \( x-y = (x-y)(-y)(-x) \) \([8,11,13]\).

### 2. The Monogeneity Symmetrical Subhypergroups

A subset \( h \) of a hypergroup \( H \) is called semisubhypergroup if \( x + y \leq h \), for every \( x, y \) in \( h \), while it is called subhypergroup of \( H \) if, for every element \( x \) of \( h \), it holds \( x + h = h + x = h \). Moreover a subhypergroup \( h \) of \( H \) is called closed from the right (in \( H \)), (resp. from the left) if, for every \( x \in FNh \), it holds \( (x+h) \cap h = \emptyset \) (resp. \( (h+x) \cap h = \emptyset \)). \( h \) is called closed if it is closed both, from the right and from the left \([3,12]\). It has been proved \([5]\) that a subhypergroup is closed if and only if it is stable under the induced
hypercompositions. Also a subhypergroup \( h \) of \( H \) is called invertible from the right (in \( H \)), (resp. from the left) if \( xh \neq yh \), with \( x, y \in H \) implies \( xh \cap yh = \emptyset \). \( h \) is called invertible if it is invertible both, from the right and from the left \([3, 12]\). From the definition it derives that every invertible subhypergroup is also closed but the opposite is not valid.

The special properties of the \( FJH \) give special types of subhypergroups that are associated to the above mentioned general types of subhypergroups \([14]\). Indeed, starting from the fact that a \( FJH \) is a join hypergroup, we find subhypergroups that are join, i.e. subhypergroups that are join hypergroups themselves. As it has been proved, the join subhypergroups of the \( FJH \) are the closed ones \([14]\). Moreover it has been proved that every join subhypergroup of a \( FJH \) is invertible and conversely \([14]\). Thus the set of the join subhypergroups of a \( FJH \) is a complete lattice, which coincides with the complete lattices of its closed and of its invertible subhypergroups. Also the fact that there exist subhypergroups of a \( FJH \) that do not contain the opposite of each one of their elements, while there exist others that they do, leads to the definition of the symmetrical subhypergroups. A subhypergroup \( h \) of a \( FJH \) is called symmetrical, if \( -x \in h \) for every \( x \in h \). It is known that the intersection of two subhypergroups is not always a subhypergroup. In the case of the symmetrical subhypergroups though, the intersection of two such subhypergroups is always a symmetrical subhypergroup. Therefore the set of the symmetrical subhypergroups of a \( FJH \) consists a complete lattice. It has been proved that the lattice of the join subhypergroups of a \( FJH \) is a sublattice of the lattice of the symmetrical subhypergroups of the \( FJH \) \([14]\).

The following paragraphs contain the study of the monogene symmetrical subhypergroup, i.e. the symmetrical subhypergroup, which is generated by a single element. So let \( (H, +) \) be a \( FJH \), let \( x \) be an arbitrary element of \( H \) and let \( h(x) \) be the monogene symmetrical subhypergroup which is generated by this element. Then, in accordance to the theory of the monogene canonical hypergroups \([17]\), and using similar symbolisms we have:

\[
\begin{cases}
  x + x + \ldots + x & \text{for } n \text{ times} \\
  0 & \text{for } n = 0 \\
  -(x) + -(x) + \ldots + -(x) & \text{for } -n \text{ times}
\end{cases}
\]

\[
(n = 0) \quad n > 0 \quad (n < 0)
\]

(1)

Next, obviously, if \( H \) is normal, then

\[
\begin{cases}
  (m+n) \cdot x & \text{if } mn > 0 \\
  (m+n) \cdot x + \min\{ |m|, |n| \} \cdot (x-x) & \text{if } mn < 0
\end{cases}
\]

(2)

On the contrary, if \( H \) is abnormal, then for some of its elements, the equality \((-n) \cdot (x-x) = n \cdot (x-x)\) is not valid. For this reason, the \( FJHs \) that are used in the following text are normal. From the above it derives that
\[(m+n)x \subseteq mx + nx\]  \hspace{1cm} (3)

For the monogene symmetrical hypergroups it holds:

**Proposition 2.1.** For every \( x \in H \):

\[ h(x) = \cup_{(m,n) \in \mathbb{Z}^2} mx + nx \cdot (x-x) \]

**Proof.** Indeed, it is known that the symmetrical subhypergroup of a normal \( FJH \) which is generated from a non-empty set \( X \) consists of the unions of all the finite sums of the elements that are contained in the union \(-X \cup X\) [14]. Thus from (1) we have:

\[ h(x) = \cup_{(k,l) \in \mathbb{N}} kx + l \cdot (-x) = \cup_{(k,l) \in \mathbb{N}} kx - l \cdot x \]

and according to (2), it is \( kx - l \cdot x = (k-l) \cdot x + \min\{k,l\} \cdot (x-x) \). So the Proposition.

**Remark 2.1.**

Since \( x \) is normal, it holds \(-x-x = x-x \), so \(-n\cdot x-x = n\cdot x-x \) and therefore, in the previous sum, we can consider that \((m,n) \in \mathbb{Z} \times \mathbb{N}\) instead of \((m,n) \in \mathbb{Z}^2\).

Since \( 0 \in x-x \), we have \( mx + nx \cdot (x-x) \subseteq mx + n' \cdot (x-x) \) for \( n < n' \)

For \( x = 0 \), it is \( h(0) = \{0\} \).

Let’s define now a symbol \( \omega(x) \) (which can even be the \(+\infty\) ), and name it order of \( x \) and simultaneously order of the monogene subhypergroup \( h(x) \). Two cases can appear such that one revokes the other:

**I.** For any \((m,n) \in \mathbb{Z} \times \mathbb{N}\), with \( m \neq 0 \), we have:

\[ 0 \not\in mx + n \cdot (x-x) \]

Then we define the order of \( x \) and of \( h(x) \) to be the infinity and we write \( \omega(x) = +\infty \).

We remark that \( 0 \not\in mx + n \cdot (x-x) \) implies that \( x \) is not an \( a \)-element, also that \( mx \) does not contain \( a \)-elements, for every \( m \in \mathbb{Z} \), and that

\[ mx \cap n \cdot (x-x) = mx \cap (n \cdot x - n \cdot x) = \emptyset \]

Yet, as it derives from (2),

\[ mx \cap (n \cdot x - n \cdot x) = \emptyset \Rightarrow \begin{cases} (m+n) x \cap n x = \emptyset \quad & \text{if} \ m > 0 \\ (n-m) x \cap n x = \emptyset \quad & \text{if} \ m < 0 \end{cases} \]
If $n=0$, then $0 \in m \cdot x$ and assuming that $m=m_1+ m_2$ with $m_1 m_2 > 0$, then:

$$\{0\} \cap m \cdot x = \emptyset \Rightarrow 0 \in m_1 \cdot x + m_2 \cdot x \Rightarrow -m_1 \cdot x \cap m_2 \cdot x = \emptyset$$

So, we generally have

$$0 \notin m \cdot x + n \cdot (x - x) \text{ for } (m, n) \in \mathbb{Z} \times \mathbb{N} \Rightarrow m \cdot x \cap m' \cdot x = \emptyset \text{ for every } m, m' \in \mathbb{Z}, \text{ with } m' \neq m$$

Conversely now, if for every $m \in \mathbb{Z} \setminus \{0\}$ the intersection $m \cdot x \cap A$ is void and if for every $m', m'' \in \mathbb{Z}$ with $m' \neq m''$ the intersection $m' \cdot x \cap m'' \cdot x$ is also void, then $0 \notin m' \cdot x - m'' \cdot x$ and therefore

$$0 \notin (m' - m'') \cdot x \text{ if } m'm'' < 0$$

and

$$0 \notin (m' - m'') \cdot x + \min\{ |m'|, |m''| \} \cdot (x - x) \text{ if } m'm'' > 0$$

Thus, generally,

$$0 \notin m \cdot x + n \cdot (x - x) \text{ for every } (m, n) \in \mathbb{Z} \times \mathbb{N}$$

Consequently, we have the Proposition:

**Proposition 2.2.** $\omega(x) = +\infty$ if and only if, for every $m \in \mathbb{Z} \setminus \{0\}$, the intersection $m \cdot x \cap A$ is void and for every $m', m'' \in \mathbb{Z}$ with $m' \neq m''$ it holds $m' \cdot x \cap m'' \cdot x = \emptyset$.

**II.** There exist $(m, n) \in \mathbb{Z} \times \mathbb{N}$ with $m \neq 0$ such that

$$0 \in m \cdot x + n \cdot (x - x)$$

Let $p$ be the minimum positive integer, such that there exists $n \in \mathbb{N}$ for which

$$0 \in p \cdot x + n \cdot (x - x)$$

**Proposition 2.3.** For a given $m \in \mathbb{Z} \setminus \{0\}$ there exist $n' \in \mathbb{N}$ such that

$$0 \in m \cdot x + n' \cdot (x - x)$$

if and only if $m$ is divided by $p$.

**Proof.** Let $m = kp$, $k \in \mathbb{Z}$. From $0 \in p \cdot x + n \cdot (x - x)$ it derives that $0 \in kp \cdot x + kn \cdot (x - x) = m \cdot x + kn \cdot (x - x)$ and therefore $n' = kn$ verifies the Proposition.

Conversely now. If $x$ is an $a$-element, then $0 \in x + n \cdot (x - x)$ for every $n \in \mathbb{N}$, so $p = 1$, and thus the Proposition. Now, let $x$ be a $c$-element, and $0 \in m \cdot x + n' \cdot (x - x)$ with $m = kp + r$, $k \in \mathbb{Z}$, $0 < r < p$. Then:
\[ m \cdot x + n' \cdot (x \cdot x) = (k \cdot p + r) \cdot x + n' \cdot (x \cdot x) \subseteq k \cdot p \cdot x + r \cdot x + n' \cdot (x \cdot x) \]

Therefore \( 0 \in k \cdot p \cdot x + r \cdot x + n' \cdot (x \cdot x) \). But \( 0 \in p \cdot x + n \cdot (x \cdot x) \), and since the sum of two non opposite \( c \)-elements does not contain any \( a \)-elements, (see the above mentioned fundamental properties of the \( FJH \)) there do not exist \( a \)-elements in \( p \cdot x \), and so

\[ -p \cdot x \subseteq -k \cdot p \cdot x + p \cdot x + n \cdot (x \cdot x) = p \cdot (x \cdot x) + n \cdot (x \cdot x) = (p + n) \cdot (x \cdot x) \]

Thus \( k \cdot p \cdot x \subseteq -k(p + n) \cdot (x \cdot x) = |k| (p + n) \cdot (x \cdot x) \) and therefore \( 0 \in p \cdot x + [|k| (p + n) + n'] \cdot (x \cdot x) \), which contradicts the supposition according to which \( p \) is the minimum positive integer with the property \( 0 \in p \cdot x + n \cdot (x \cdot x) \). Thus \( r = 0 \), and so \( m = k \cdot p \).

For \( m = k \cdot p \) \((k \in \mathbb{Z})\), and for \( q(k) \) being the minimum non negative integer for which \( 0 \in k \cdot p \cdot x + q(k) \cdot (x \cdot x) \), we define a function \( q: \mathbb{Z} \to \mathbb{N} \) such that it corresponds \( k \) to \( q(k) \). We call order of \( x \) and of \( h(x) \) the pair \( \omega(x) = (p, q) \), principal order -of \( x \) and of \( h(x) \)- the number \( p \) and associated order -of \( x \) and of \( h(x) \)- the function \( q \).

Consequently, according to the above, if \( x \) is an \( a \)-element, then \( 0 \in x + (x \cdot x) \) and therefore \( \omega(x) = (1, q) \) with \( q(k) = 1 \) for every \( k \in \mathbb{Z} \) \( \setminus \{0\} \). Moreover, if \( x \) is a selfopposite \( c \)-element, then \( 0 \in 2 \cdot x + 0 \cdot (x \cdot x) \), if \( x \in x \cdot x \) and \( 0 \in x + (x \cdot x) \), if \( x \in x - x \) and thus \( \omega(x) = (2, q) \) with \( q(k) = 0 \) in the first case and \( \omega(x) = (1, q) \) with \( q(k) = 1 \) in the second case (for every \( k \in \mathbb{Z} \)).

Also we remark that the order of \( 0 \) is \( \omega(0) = (1, q) \), with \( q(k) = 0 \), for every \( k \in \mathbb{Z} \), since \( 0 \in 1 \cdot 0 + 0 \cdot (0 \cdot 0) \) and \( 0 \) is the only element which has this property. Yet, it is possible that there exist elements \( x \in H \), \( x \neq 0 \) with prime order 1, and for this thing to happen it is necessary and sufficient that there exists an integer \( n \) such that \( -x \in n \cdot (x \cdot x) \) [as for instance in the above second case of the self opposite \( c \)-elements].

Similarly to the monogene symmetrical hypergroups, we have the monogene join ones. It is known that the join subhypergroup \( X \) of a \( FJH \), which is generated from a non empty set \( X \), consists of all the \( a \)-elements as well as of the unions of all the finite sums of the \( c \)-elements that are contained in the union \( -X \cup X \) \([14]\). Thus, we reach analogous to the above results for the monogene join sub-hypergroups as well. Moreover, taking into consideration that the set \( A^\wedge \) of the \( a \)-elements and the zero element is a join subhypergroup \([14]\), yet, that this subhypergroup is the minimum in the sense of inclusion, join subhypergroup of a \( FJH \) \([14]\), and also, that the minimum join subhypergroup \( A^\wedge \) equals to \( x \cdot x \), for every \( x \in A^\wedge \) \([14]\), we end up to the following Proposition:

**Proposition 2.4.** In every \( FJH \) the monogene symmetrical subhypergroup with generator one of its \( a \)-elements \( x \) is:

\[ S(x) = \bigcup_{n \in \mathbb{Z}} n \cdot (x \cdot x) \]

and the monogene join is:

\[ J(x) = A^\wedge = A \cup \{0\} \]
Remark 2.2.
$A^\wedge$ is monogene join with generator each one of its elements.

If $x$ is a $c$-element, then $x \cdot \psi = x - \psi$ (for every $\psi$ from the $FJH$) [13] and moreover the difference $x - x$ contains all the $a$-elements [13]. Taking into account that a necessary and sufficient condition for a non empty subset $h$ of a $FJH H$ to be a join subhypergroup of $H$, is that $x \cdot \psi \subseteq h$ and $x - \psi \subseteq h$ for every $x \cdot \psi \in h$ [14], we have the

**Proposition 2.5.** In every $FJH$ the monogene symmetrical subhypergroup with generator one of its $c$-elements is join, that is:

$$J(x) = S(x) = \cup_{(m,n) \in \mathbb{Z}} [m \cdot x + n \cdot (x - x)]$$

and moreover: $A^\wedge \subseteq J(x)$.

**Bibliography**


