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ON CERTAIN FUNDAMENTAL PROPERTIES of HYPERGROUPS and FUZZY HYPERGROUPS – MIMIC FUZZY HYPERGROUPS

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ABSTRACT

In this paper, properties of hypergroups pertaining to the relationship between the hypercomposition and induced hypercompositions with the empty set are presented. Analogous fuzzy hypergroup properties are also proven. Finally, the study of these properties leads to the introduction of the mimic fuzzy hypergroup (fuzzy - hypergroup).

KEYWORDS

hypergroup, fuzzy hypergroup.

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1. INTRODUCTION

Almost all fields of science desire to and attempt to utilize mathematical models in the process of not only describing the various phenomena under study, but in also predicting, wherever and whenever possible, the results generated through the influence of various causes. Several of the mathematical models thus utilized are non-deterministic, as several phenomena incorporate numerous uncertainties. Theory of probability, stochastic processes, fuzzy set theory, soft set theory, vague set theory, hypercompositional algebra, etc., are all different ways of expressing uncertainty. The self-contained development, as well as the cross-linking of the above theories hold great attraction for mathematicians. For example, A. Maturo, in a series of articles of his, (e.g. [39, 40, 41, 42] delved into linking the theory of probability with hypercompositional algebra. On the other hand, several mathematicians worked on linking fuzzy set theory with hypercompositional algebra.

One can distinguish three approaches, which were employed in order to connect these two topics. One approach is to consider a certain hyperoperation defined through a fuzzy set (P. Corsini [4], P. Corsini - V. Leoreanu, [6], I. Cristea e.g. [7, 8, 9], I. Cristea - S. Hoskova [10], M. Stefanescu - I. Cristea [49], K. Serafimidis et al. [48] etc.). Another is to consider fuzzy hyperstructures in a similar way as Rosenfeld did for fuzzy groups [46] (Zahedi, A...
Hasankhani [12, 54, 55], B. Davvaz [11] and others). The third approach is employed in the pioneering papers by P. Corsini - I. Tofan [5] and by I. Tofan - A. C. Volf [50, 51], which introduce fuzzy hyperoperations that induce fuzzy hypergroups. This approach was further adopted by other researchers (Ath. Kehagias e.g. [15, 16, 17, 18], V. Leoreanu-Fotea e.g. [22, 23] etc.). M. K. Sen, R. Ameri and G. Chowdhury utilized this concept in defining fuzzy hypersemigroups [47].

This paper deals with the algebra of hypergroups and fuzzy hypergroups. In researching all relevant bibliography, we came to realize that there exists a certain amount of confusion in some fundamental matters. For example, in attempting to define the hypergroup, some authors view the hyperoperation on a non-empty set $H$ as a function from $H \times H$ to the power set $P(H)$ of $H$, while others view it as a function from $H \times H$ to $P^*(H)$, i.e. to the set of all non-empty subsets of $H$. Certain fundamental properties of hypergroups and fuzzy hypergroups, which are proven herein, afford us a clearer view of such matters.

2. ON CRISP HYPERCOMPOSITIONS

Hypercompositional algebra was born in 1934, when F. Marty, in order to study problems in non-commutative algebra, such as cosets determined by non-invariant subgroups, generalized the notion of the group, thus defining the hypergroup [24]. The hypergroup is an algebraic structure in which the result of the composition of two elements is not an element but a set of elements. To make this paper self-contained, we begin by listing some definitions from the theory of hypercompositional structures (see also [35]). A (crisp) hypercomposition or hyperoperation in a non-empty set $H$ is a function from $H \times H$ to the power set $P(H)$ of $H$. A non-empty set $H$ endowed with a hypercomposition “$\bullet$” is called hypergroupoid if $ab \neq \emptyset$ for any $a, b \in H$, otherwise it is called partial hypergroupoid. Note that, if $A, B$ are subsets of $H$, then $AB$ signifies the union $\bigcup_{(a,b) \in A \times B} ab$. Since $A \times B = \emptyset \Leftrightarrow A = \emptyset$ or $B = \emptyset$, one can observe that if $A = \emptyset$ or $B = \emptyset$, then $AB = \emptyset$ and vice versa. $aA$ and $Aa$ have the same meaning as $\{a\}A$ and $A\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified with its member $a$.

A hypergroup is a non-empty set $H$ endowed with a hypercomposition which satisfies the following axioms:

i. $(ab)c = a(bc)$ for every $a, b, c \in H$ (associativity) and

ii. $aH = Ha = H$ for every $a \in H$ (reproduction).

If only (i) is valid, then the hypercompositional structure is called semi-hypergroup, while if only (ii) is valid, then it is called quasi-hypergroup. The quasi-hypergroups in which the weak associativity is valid, i.e. $(ab)c \cap a(bc) \neq \emptyset$ for every $a, b, c \in H$, were named $H_v$-groups [52].

Remark. If a non-empty set $H$ is endowed with a composition which satisfies the
associative and the reproduction axioms, then $H$ is a group. Indeed let $x \in H$. Per reproduction $x \in xH$. Therefore there exists $e \in H$ such that $xe = x$. Next let $y$ be an arbitrary element in $H$. Per reproduction there exists $z \in H$ such that $y = zx$. Consequently $ye = (zx)e = z(xe) = zx = y$. Hence $e$ is a right neutral element. Now, per reproduction $e \in xH$. Thus there exists $x' \in H$, such that $e = xx'$. Hence any element in $H$ has a right inverse.

Proposition 2.1. The result of the hypercomposition of any two elements in a hypergroup is always non-void.

Proof. Let $H$ be a hypergroup and suppose that $ab = \emptyset$ for some $a, b \in H$. Per reproduction, $aH = H$ and $bH = H$. Hence, $H = aH = a(bH) = (ab)H = \emptyset$, which is absurd.

Proposition 2.2. If the weak associativity is valid in a hypercompositional structure, then the result of the hypercomposition of any two elements is always non-void.

Proof. Let $H$ be a non-void set endowed with a hypercomposition satisfying the weak associativity. Suppose that $ab = \emptyset$ for some $a, b \in H$. Then, $(ab)c = \emptyset$ for any $c \in H$. Therefore, $(ab)c \cap a(bc) = \emptyset$, which is absurd. Hence, $ab$ is non-void.

Corollary 2.1. The result of the hypercomposition of any two elements in a $HV$-group is always a non-void set.

F. Marty also defined in [24] the two induced hypercompositions (right and left division) resulting from the hypercomposition of the hypergroup, i.e.: 

$$\begin{align*}
\frac{a}{b} &= \{x \in H \mid a \in xb\} \quad \text{and} \quad \frac{a}{b} = \{x \in H \mid a \in bx\}.
\end{align*}$$

It is obvious that, if the hypergroup is commutative, then the two induced hypercompositions coincide. For the sake of notational simplicity, $a/b$ or $a : b$ is used to denote the right division (as well as the division in commutative hypergroups) and $b \setminus a$ or $a . b$ is used to denote the left division. F. Marty's life was cut short, as he was killed during a military mission in World War II. [24, 25, 26] are the only works on hypergroups he left behind. However, several relevant papers by other authors began appearing shortly thereafter (e.g. Krasner [19, 20], Kuntzmann [21] etc). Up to the present, a vast number of papers has been produced on this subject (e.g.: see [3, 6])

In [13] and then in [14], a principle of duality is established in the theory of hypergroups. More precisely, two statements of the theory of hypergroups are dual, if each results from the other by interchanging the order of the hypercomposition, i.e. by interchanging any hypercomposition $ab$ with the hypercomposition $ba$. One can observe that the associativity axiom is self-dual. The left and right divisions have dual definitions, thus they must be interchanged in a construction of a dual statement. Therefore, the following principle of duality holds true:
Given a theorem, the dual statement resulting from interchanging the order of hypercomposition “.” (and, by necessity, interchanging of the left and the right divisions), is also a theorem.

This principle is used throughout this paper. The following properties are direct consequences of the hypergroup axioms and the principle of duality is used in their proofs.

Proposition 2.3. \( a/b \neq \emptyset \) and \( b \setminus a \neq \emptyset \) for all the elements \( a, b \) of a quasi-hypergroup \( H \).

Proof. Per reproduction, \( Hb = H \) for every \( b \in H \). Hence, for every \( a \in H \) there exists \( x \in H \), such that \( a \in xb \). Thus, \( x \in a/b \) and, therefore, \( a/b \neq \emptyset \). Dually, \( b \setminus a \neq \emptyset \).

Proposition 2.4. In a quasi-hypergroup \( H \), the non-empty result of the induced hypercompositions is equivalent to the reproduction axiom.

Proof. Suppose that \( x/a \neq \emptyset \) for every \( a, x \in H \). Thus, there exists \( y \in H \), such that \( x \in ya \). Therefore, \( x \in Ha \) for every \( x \in H \) and so \( H \subseteq Ha \). Next, since \( Ha \subseteq H \) for every \( a \in H \), it follows that \( H = Ha \). Per duality, \( H = aH \). Conversely now, per Proposition 2.3, the reproduction axiom implies that \( a/b \neq \emptyset \) and that \( a \setminus b \neq \emptyset \) for every \( a, b \) in \( H \).

Based on Proposition 2.4, we are now in a position to give an equivalent definition of the hypergroup.

Definition 2.1. A hypergroup is a non-empty (crisp) set \( H \) endowed with a (crisp) hypercomposition, i.e. a function from \( H \times H \) to the powerset \( P(H) \) of \( H \), which satisfies the following axioms:

i. \( (ab)c = a(bc) \) for every \( a, b, c \in H \) (associativity) and

ii. \( a/b \neq \emptyset \) and \( b \setminus a \neq \emptyset \) for every \( a, b \in H \).

Proposition 2.5. In a hypergroup \( H \), equalities (i) \( H = H/a = a/H \) and (ii) \( H = a\setminus H = H \setminus a \) are valid for every \( a \) in \( H \).

Proof. (i) Per Proposition 2.1, the result of hypercomposition in \( H \) is always a non-empty set. Thus, for every \( x \in H \) there exists \( y \in H \), such that \( y \in xa \), which implies that \( x \in y/a \). Hence, \( H \subseteq H/a \). Moreover, \( H/a \subseteq H \). Therefore, \( H = H/a \). Next, let \( x \in H \). Since \( H = xH \), there exists \( y \in H \) such that \( a \in xy \), which implies that \( x \in a/y \). Hence, \( H \subseteq a/H \). Moreover, \( a/H \subseteq H \). Therefore, \( H = a/H \). (ii) follows by duality.

The hypergroup is a very general structure, which was progressively enriched with additional axioms, either more or less powerful. This created a significant number of specific hypergroups. Moreover, some of these hypergroups constituted a constructive origin for the development of other new hypercompositional structures (e.g.: see [3, 6, 27, 28, 29, 32, 34, 36, 43]). Thus, W. Prenowitz enriched hypergroups with an axiom, in order to utilize them in the study of geometry [e.g.: see 44, 45]. More precisely, he introduced into the commutative hypergroup the transposition axiom

\[ a/b \cap c/d \neq \emptyset \implies ad \cap bc \neq \emptyset \] for every \( a, b, c, d \in H \)
and named this new hypergroup join space \[44\]. For the sake of terminology unification, join spaces are also called join hypergroups \[30\]. It has been proven that these hypergroups also comprise a useful tool in the study of languages and automata \[34, 37\]. Later on, J. Jantosciak generalized the above axiom in an arbitrary hypergroup as follows:

\[
\{ b \setminus a \cap c \} \cap d \neq \emptyset \quad \text{implies} \quad \{ ad \} \cap \{ bc \} \neq \emptyset \quad \text{for every} \quad a, b, c, d \in H.
\]

He named this particular hypergroup transposition hypergroup \[13\]. Clearly, if \( A, B, C \) and \( D \) are subsets of \( H \), then \( \{ B \setminus A \cap C \} \cap D \neq \emptyset \) implies that \( \{ AD \} \cap \{ BC \} \neq \emptyset \). In \[31, 38\] specialized transposition hypergroups were studied and, in \[33\], the transposition axiom was introduced into \( \mathcal{H}_V \)-groups and the transposition \( \mathcal{H}_V \)-group was thus defined.

3. ON FUZZY HYPERCOMPOSITIONS

Zadeh, in 1965, in order to provide «a natural way of dealing with problems in which the source of imprecision is the absence of sharply defined criteria of class membership, rather than the presence of random variables» \[53\] introduced the notion of fuzzy sets. If \( H \) is a non-void crisp set, then a fuzzy subset of \( H \) is a mapping from \( H \) to the interval of real numbers \([0,1]\). If \( A \subseteq H \), then the characteristic function \( X_A \) of \( A \)

\[
X_A : H \to [0,1], \quad x \mapsto X_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}
\]

is a fuzzy subset of \( H \). Thus, if \( A = \emptyset \), then its characteristic function is the identically zero function \( 0_H \), i.e. \( X_\emptyset (x) = 0_H (x) = 0 \) for every \( x \in H \). Moreover, if \( A = H \), \( H \neq \emptyset \), then the characteristic function of the entire set \( H \), is \( X_H(x) = 1_H(x) = 1 \) for every \( x \in H \). Thus, we can consider crisp sets as special case of fuzzy sets and identify every set (crisp or fuzzy) with its membership function. The collection of all fuzzy subsets of \( H \) is denoted by \( F(H) \).

A fuzzy hypercomposition maps the pairs of elements of the Cartesian product \( H \times H \) to fuzzy subsets of \( H \), i.e. \( \circ : H \times H \to F(H) \). Hence, if \( \circ \) is a fuzzy hyperoperation, then \( a \circ b \) is a function and the notation \( (a \circ b)(x) \) means the value of \( a \circ b \) at the element \( x \). The definition of the fuzzy hyperoperation subsumes the relevant one of crisp hyperoperation as a special case, since the later results from the former using the characteristic function.

Definition 3.1. \[17, 18\] If \( \circ : H \times H \to F(H) \) is a fuzzy hypercomposition, then for every \( a \in H, b \in F(H) \), the fuzzy sets \( a \circ B \) and \( B \circ a \) are defined respectively by

\[
(a \circ B)(z) = \vee_{y \in H} \left( \left[ (a \circ y)(z) \right] \land B(y) \right)
\]

\[
(B \circ a)(z) = \vee_{y \in H} \left( \left[ (y \circ a)(z) \right] \land B(y) \right).
\]

Per definition 3.1, if \( a, b, c \in H \), then:
\[(a \circ (b \circ c))(z) = \lor_{y \in H} \left[ (a \circ y)(z) \land (b \circ c)(y) \right] \]

and

\[
((a \circ b) \circ c)(z) = \lor_{y \in H} \left[ (y \circ c)(z) \land (a \circ b)(y) \right] 
\]

Definition 3.2. [17, 18] If \( \circ : H \times H \rightarrow \mathcal{F}(H) \) is a fuzzy hypercomposition, then, for every \( A, B \in \mathcal{F}(H) \), the fuzzy set \( A \circ B \) is defined by

\[
(A \circ B)(z) = \lor_{x,y \in H} \left[ \left[ (x \circ y)(z) \right] \land A(x) \land B(y) \right].
\]

As mentioned above, these definitions subsume the relevant ones of crisp hyperoperations as special cases. For example, if \( X_A \) is the characteristic function of the crisp set \( A \), then, per Definition 2.1:

\[
(x \circ X_A)(z) = \lor_{y \in H} \left[ \left[ (x \circ y)(z) \right] \land X_A(y) \right] = \lor_{\{y \in \mathcal{F}(H) : x \circ y \neq \emptyset\}} X_A(y).
\]

Hence \( x \circ X_A \) is the characteristic function of the crisp set \( xA = \cup_{y \in \mathcal{F}(H)} xy \) (see also [17]).

Definition 3.3. If \( a \in H \) and \( A, B \in \mathcal{P}(H) \), then \( a \circ B = a \circ X_B \), \( B \circ a = X_B \circ a \) and \( A \circ B = X_A \circ X_B \).

Definition 3.4. [5, 51] If \( \circ : H \times H \rightarrow \mathcal{F}(H) \) is a fuzzy hypercomposition, then \( H \) is called fuzzy hypergroup, if the following two axioms are valid:

i. \( (a \circ b) \circ c = a \circ (b \circ c) \) for every \( a, b, c \in H \) (associativity),

ii. \( a \circ H = H \circ a = X_H \) for every \( a \in H \) (reproduction).

If (i) is only valid, then \( H \) is called a fuzzy semi-hypergroup [47] while if (ii) is only valid, then \( H \) is called a fuzzy quasi-hypergroup.

Lemma 3.1. For every \( a, b \in H \) and \( C \in \mathcal{F}(H) \), the following is true:

\[
(a \circ b) \circ C = a \circ (b \circ C).
\]

Proof. \[
[(a \circ b) \circ C](z) = \lor_{x,y \in H} \left[ (y \circ x)(z) \land (a \circ b)(y) \land C(x) \right] =
\]

\[
= \lor_{x \in H} \left[ \lor_{y \in H} \left[ (y \circ x)(z) \land (a \circ b)(y) \right] \land C(x) \right] =
\]

\[
= \lor_{x \in H} \left[ (a \circ b)(x)(z) \land C(x) \right] = \lor_{x \in H} \left[ (a \circ b)(x)(z) \land C(x) \right] =
\]

\[
= \lor_{x \in H} \left[ (a \circ y)(z) \land (b \circ x)(y) \land C(x) \right] =
\]

\[
= \lor_{x,y \in H} \left[ (a \circ y)(z) \land (b \circ x)(y) \land C(x) \right] =
\]

\[
= \lor_{x,y \in H} \left[ (a \circ y)(z) \land (b \circ x)(y) \land C(x) \right] =
\]
\begin{align*}
&= \forall y \in H \left[ (a \circ y)(z) \land \left[ \forall z \in H \left( b \circ x \right)(y) \land C(x) \right] \right] = \\
&= \forall y \in H \left[ (a \circ y)(z) \land (b \circ C)(y) \right] = \left[ a \circ (b \circ C) \right](z).
\end{align*}

Proposition 3.1. $a \circ b \neq 0_H$ is valid for any pair of elements $a, b$ in a fuzzy hypergroup $H$.

Proof. Suppose that $a \circ b = 0_H$ for some $a, b \in H$. Per reproduction, $a \circ H = X_H$ and $b \circ H = X_H$. Hence, $X_H = a \circ H = a \circ (b \circ H)$. Per Lemma 3.1, the equality $a \circ (b \circ H) = (a \circ b) \circ H$ is valid. Since $a \circ b = 0_H$, we have: $(a \circ b) \circ H = 0_H \circ X_H$. But $(0_H \circ X_H)(z) = \forall x, y \in H \left[ (a \circ y)(z) \right] \land 0_H \circ X_H(y) = 0_H(z)$. Therefore, $X_H = 0_H$, which is absurd because $H \neq \emptyset$.

The fuzzy $H_V$-groups were defined in [15]. If $A, B \in F(H)$ and $p \in (0, 1]$, then we write $A \boxdot_p B$, if there exists $x \in H$ such that $A(x) \land B(x) \geq p$. The fuzzy quasi-hypergroups in which the weak associativity is valid, i.e. $(a \circ b) \circ c \boxdot_p a \circ (b \circ c)$ for every $a, b, c \in H$, are named fuzzy $H_V$-groups.

Proposition 3.2. If a fuzzy hypercompositional structure $(H, \circ)$ is endowed with the weak associativity, then $a \circ b \neq 0_H$ is valid for every $a, b \in H$.

Proof. Suppose that $a \circ b = 0_H$ for some $a, b \in H$. Then $(a \circ b) \circ c = 0_H \circ c$, for any $c \in H$. Since $(0_H \circ c)(z) = \forall y \in H \left[ (y \circ c)(z) \right] \land 0_H(y) = 0$, it follows that $0_H \circ c = 0_H$ and, therefore, $(a \circ b) \circ c = 0_H$. Hence, the weak associativity is not valid in $(H, \circ)$, which contradicts our supposition.

Corollary 3.1. The result of the hypercomposition of any two elements in a fuzzy $H_V$-group is always a non-zero function.

4. THE MIMIC FUZZY HYPERGROUP

If $H$ is a non-void set endowed with a fuzzy hypercomposition $\circ$, then two new induced fuzzy hypercompositions “$/$” and “$\backslash$” can be defined as follows:

\[
(a \div b)(x) = (x \circ b)(a) \quad \text{for every } a, b, x \in H
\]

\[
(b \backslash a)(x) = (b \circ x)(a) \quad \text{for every } a, b, x \in H.
\]

As in the case of crisp hypercompositions, the two induced fuzzy hypercompositions will be called fuzzy right division and fuzzy left division respectively (see also [2]).
Proposition 4.1. For any pair of elements \(a, b\) in a fuzzy hypergroup \(H\), \(a / b \neq 0_H\) and \(a \setminus b \neq 0_H\) is valid.

Proof. Per reproduction, \(H \circ b = X_H\) is valid for every \(b \in H\). Thus, equality \((H \circ b)(a) = X_H(a)\) is true for any \(a \in H\). Since

\[
(H \circ b)(a) = \lor_{y \in H} [(y \circ b)(a) \land X_H(y)],
\]

it follows that there exists \(y \in H\) such that \((y \circ b)(a) = 1\) or, equivalently, \((a / b)(y) = 1\). Therefore, \(a / b \neq 0_H\). Dually, \(a \setminus b \neq 0_H\).

It becomes obvious that a statement analogous to Proposition 2.4 is not valid in the case of fuzzy hypergroups. Indeed:

Example 4.1. Let \(H = \{a, b\}\) and suppose that

\[
(a \circ a)(a) \leq (a \circ a)(b) \leq (a \circ b)(a) \leq (b \circ b)(a)
\]

and

\[
(a \circ a)(b) = (a \circ b)(b) = (b \circ a)(a) = (b \circ a)(b) \leq (b \circ b)(a) \leq (b \circ b)(b)
\]

Then, \((H, \circ)\) is a fuzzy semi-hypergroup. Suppose that \((x \circ y)(z) \neq 0\) for every \(x, y, z \in H\). Then, in this fuzzy semi-hypergroup, \(a / b \neq 0_H\) and \(a \setminus b \neq 0_H\) is valid for every \(a, b \in H\). Yet, if \((x \circ y)(z) = 1\) for any \(x, y, z \in H\), one can easily see that the reproduction is not verified.

These ideas lead to the introduction of the following definition:

Definition 4.1. If \(\circ : H \times H \to F(H)\) is a fuzzy hypercomposition, then \(H\) is called mimic fuzzy hypergroup (fuzzy \(M\)-hypergroup), if the following two axioms are valid:

i. \((a \circ b) \circ c = a \circ (b \circ c)\) for every \(a, b, c \in H\) (associativity),

ii. \(a / b \neq 0_H\) and \(a \setminus b \neq 0_H\) for every \(a, b \in H\).

If (ii) is only valid, then \(H\) is called mimic fuzzy quasi-hypergroup (fuzzy \(M\)-quasi-hypergroup) while, if instead of (i) the weak associativity is valid, then \(H\) is called mimic fuzzy \(H\)-group (fuzzy \(M_{H^{\circ}}\)-group).

Proposition 4.2. In a fuzzy \(M\)-hypergroup \(H\), it holds that \((H \circ a)(x) \neq 0\) and \((a \circ H)(x) \neq 0\) for every \(a, x \in H\).

Proof. Since \(x / a \neq 0_H\) for every \(a, x \in H\), it follows that there exists \(y \in H\) such that \((x / a)(y) \neq 0\). Hence, there exists \(y \in H\) such that \((y \circ a)(x) \neq 0\), for any \(a, x \in H\). Since
\[(H \circ a)(x) = \bigvee_{y \in H} \left( \left[ (y \circ a)(x) \right] \wedge X_H(y) \right), \] it follows that \((H \circ a)(x) \neq 0\) for every \(a, x \in H\).

Per duality, \((a \circ H)(x) \neq 0\) for every \(a, x \in H\).

**Proposition 4.3.** In a fuzzyM-hypergroup \(H\), it holds that \(a \circ b \neq 0_H\) for every \(a, b \in H\).

**Proof.** Suppose that there are \(a, b \in H\) such that \(a \circ b = 0_H\). Then \((a \circ b) \circ H = 0_H\). Per Lemma 3.1, \((a \circ b) \circ H = a \circ (b \circ H)\). Hence \(a \circ (b \circ H) = 0_H\). But

\[\left[ a \circ (b \circ H) \right](z) = \bigvee_{y \in H} \left[ (a \circ y)(z) \wedge (b \circ H)(y) \right]\]

Per Proposition 4.2, \((b \circ H)(y) \neq 0\), for every \(b, y \in H\). Therefore \((a \circ y)(z) = 0\) for every \(z, y \in H\). Thus \((a \setminus z)(y) = 0\) for every \(y \in H\), which is absurd.

It is obvious that if a fuzzyM-hypergroup \(H\) is commutative, then \(a \circ H = H \circ a\) for any \(a \in H\). However, generally speaking, this equality is not valid. Hence, we have the following definition:

**Definition 4.2.** A fuzzyM-hypergroup \(H\) will be called commutable fuzzyM-hypergroup, if \(a \circ H = H \circ a\) for any \(a \in H\).

**Example 4.2.** Let \(H = \{a, b\}\) and suppose that:

\[
\begin{align*}
(a \circ a)(a) &= 0.1; \quad (a \circ a)(b) = 0.2 \\
(b \circ a)(a) &= (a \circ b)(b) = (b \circ a)(b) = 0.2; \quad (a \circ b)(a) = 0.5 \\
(b \circ b)(a) &= 0.7; \quad (b \circ b)(b) = 0.9
\end{align*}
\]

Then \((H, \circ)\) is a non-commutative fuzzyM-hypergroup, since \((a \circ b)(a) \neq (b \circ a)(a)\). Furthermore \((H, \circ)\) is non-commutable. Next if we define:

\[
\begin{align*}
(a \ast a)(b) &= (b \ast a)(a) = 0.1 \\
(a \ast a)(a) &= (b \ast b)(a) = (b \ast b)(b) = (a \ast b)(b) = (a \ast a)(b) = 0.9
\end{align*}
\]

then \((H, \ast)\) is a non-commutative fuzzyM-hypergroup, since \((a \ast b)(a) \neq (b \ast a)(a)\). However \((H, \ast)\) is commutable. Moreover if we define:

\[
\begin{align*}
(a \cdot a)(a) &= (b \cdot b)(a) = (a \cdot b)(a) = (b \cdot a)(a) = 0.1 \\
(a \cdot a)(b) &= (b \cdot b)(b) = (a \cdot b)(b) = (b \cdot a)(b) = 0.2
\end{align*}
\]

then \((H, \cdot)\) is a commutative fuzzyM-hypergroup.

As in the case of fuzzy hypergroups [1, 2, 15, 16], the transposition axiom can be
introduced in fuzzy\textsubscript{M}-hypergroups as well. Thus, we have the definition:

Definition 4.3. A fuzzy\textsubscript{M}-hypergroup $H$ will be called transposition fuzzy\textsubscript{M}-hypergroup, if for any $a$, $b$, $c$, $d \in H$ for which there exists $p \in (0,1]$ such that $b \triangleleft a \triangleright p \ c / d$, there exists also $q \in (0,1]$ such that $a \triangleright d \triangleleft q \ b \triangleright c$. If $p = q$ for every $a$, $b$, $c$, $d \in H$, then $H$ will be called $p$-transposition fuzzy\textsubscript{M}-hypergroup.

REFERENCES


